

THRESHOLD PHENOMENON FOR A FAMILY OF THE GENERALIZED GENERALIZED FRIEDRICHS MODELS WITH THE PERTURBATION OF RANK ONE

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ABSTRACT. A family $H_\mu(p)$, $\mu > 0$, $p \in \mathbb{T}^3$ of the Generalized Friedrichs models with the perturbation of rank one, associated to a system of two particles, moving on the three dimensional lattice \mathbb{Z}^3 , is considered. The existence or absence of the unique eigenvalue of the operator $H_\mu(p)$ lying outside the essential spectrum, depending on the values of $\mu > 0$ and $p \in U_\delta(p_0) \subset \mathbb{T}^3$ is proven. Moreover, the analyticity of associated eigenfunction is shown.

INTRODUCTION

In celebrated work [8] of B.Simon and M.Klaus it is considered a family of Schrödinger operators $H = -\Delta + \mu V$ and, a situation where as μ tends to μ_0 some eigenvalue $e_i(\mu)$ tends to 0, i.e., as μ tends to μ_0 an eigenvalue is absorbed into continuous spectrum, and conversely, as μ tends to $\mu_0 + \varepsilon$, $\varepsilon > 0$ continuous spectrum *gives birth* to a new eigenvalue. This phenomenon in [8] is called *coupling constant threshold*.

In [3] for a wide class of two-body energy operators $H_2(k)$ on the d -dimensional lattice \mathbb{Z}^d , $d \geq 3$, k being the two-particle quasi-momentum, it is proven that if the following two assumptions (i) and (ii) are satisfied, then for all nontrivial values k , $k \neq 0$, the discrete spectrum of $h(k)$ below its threshold is non-empty. The assumptions are: (i) the two-particle Hamiltonian $H_2(k)$ associated to the zero value of the quasi-momentum has either an eigenvalue or a virtual level at the bottom of its essential spectrum and (ii) the one-particle free Hamiltonians in the coordinate representation generate positivity preserving semi-groups.

In [13] the Hamiltonian of a system of two identical quantum mechanical particles (bosons) moving on the d -dimensional lattice \mathbb{Z}^d , $d \geq 3$ and interacting via zero-range repulsive pair potentials is considered. For the associated two-particle Schrödinger operator $H_\mu(K)$, $K \in \mathbb{T}^d = (-\pi, \pi]^d$ there existence of coupling constant threshold $\mu = \mu_0(K) > 0$ is proven: the operator has non eigenvalue for any $0 < \mu < \mu_0$, but for each $\mu > \mu_0$ it has a unique eigenvalue $z(\mu, K)$ above the upper edge of the essential spectrum of $H_\mu(K)$. Moreover asymptotics for $z(\mu, K)$ are found, when μ approaches to $\mu_0(K)$ and $K \rightarrow 0$.

Notice that in [8] the existence of a coupling constant threshold has been assumed, at the same time in [13] the coupling constant threshold is definitely found by the given data of the Hamiltonian.

Notice also that for the Hamiltonians of a system of two identical particles moving on \mathbb{R}^2 or \mathbb{Z}^2 the coupling constant threshold vanishes, if particles are bosons and the coupling constant threshold is positive, if particles are fermions.

In the present paper, a family of Generalized Friedrichs models under rank one perturbations $H_\mu(p)$, $\mu > 0$, $p \in U_\delta(p_0) \subset \mathbb{T}^3$, where $U_\delta(p_0)$ is a δ -neighborhood of the point

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$p = p_0 \in \mathbb{T}^3$, associated to a system of two particles on the three-dimensional lattice \mathbb{Z}^3 interacting via pair local *repulsive potentials* is considered.

If parameters of the Generalized Friedrichs model satisfy some conditions then there exists a *coupling constant threshold* $\mu = \mu(p) > 0$ that the operator has non eigenvalue for any $0 < \mu < \mu(p)$, but for any $\mu > \mu(p)$ there is a unique eigenvalue $z(\mu, p)$ of $H_\mu(p)$, which lie above the threshold $z = M(p)$ of the operator $H_\mu(p)$, $p \in U_\delta(p_0)$. For the associated eigenfunction an explicit expression is found and its analyticity is proven.

We have found necessary and sufficient conditions, in order to the threshold $z = M(p)$ was an eigenvalue or a resonance (virtual level) or a regular point of the essential spectrum of $H_\mu(p)$, $p \in U_\delta(p_0)$.

One of the reasons to consider the family of the Generalized Friedrichs models interacting via pair local *repulsive potentials* is as follows: the family of the Generalized Friedrichs models generalizes and involves some important behaviors as of the Shrödinger operators associated to the Hamiltonians for systems of two arbitrary particles moving on \mathbb{R}^d or \mathbb{Z}^d , $d \geq 1$, as well as, the Hamiltonians for systems of both bosons and fermions [11], [14, 15], [16].

Furthermore, as have been stated in [5, 21] that throughout physics stable composite objects are usually formed by way of attractive forces, which allow the constituents to lower their energy by binding together. The repulsive forces separates particles in free space. However, in structured environment such as a periodic potential and in the absence of dissipation, stable composite objects can exist even for repulsive interactions.

The family of the Generalized Friedrichs models theoretically adequately describes this phenomenon relating to *repulsive forces*, since the two-particle discrete Schrödinger operators are the special case of this family.

The Generalized Friedrichs model, i.e., the case, where the non-perturbed operator H_0 is a multiplication operator by arbitrary function with Van Hove singularities(critical points) defined on the closed interval $[a, b]$ has been considered in [9]. In this case, the multiplicity of continuous spectrum is not constant.

Generalized Friedrichs model with given number of eigenvalues embedded in the continuous spectrum has been constructed [1].

The Generalized Friedrichs models appear mostly in the problems of solid state physics [18, 19], quantum mechanics [6], and quantum field theory [7, 17] and in general settings have been studied in [14, 15].

In [2] the family of Generalized Friedrichs models under rank one perturbations $H_\mu(p)$, $\mu > 0$, $p \in (-p, p]^3$, associated to a system of two particles on the three-dimensional lattice \mathbb{Z}^3 is considered. In some special case of multiplication operator and under the assumption that the operator $H_\mu(0)$, $0 \in \mathbb{T}^3$ has a *coupling constant threshold* $\mu_0(0) > 0$ the existence of a unique eigenvalue below the bottom of the essential spectrum of $H_{\mu_0(0)}(p)$, $p \in (-p, p]^3$ for all non-trivial values of $p \in \mathbb{T}^3$ has been proved.

In [11] for a family of the Generalized Friedrichs models $H_\mu(p)$, $\mu > 0$, $p \in \mathbb{T}^2$ either the existence or absence of a positive coupling constant threshold $\mu = \mu(p) > 0$ depending on the parameters of the model has been proved.

In [12] it is established an expansion of the threshold eigenvalue $E(\mu, p)$ and resonance in some neighborhood of the point $\mu = \mu(p)$.

In [16] a special family of the Generalized Friedrichs models has been considered and the existence of eigenvalues for some values of quasi momentum $p \in \mathbb{T}^d$ of the system, lying in a neighborhood of some $p_0 \in \mathbb{T}^d$, has been proved.

1. PRELIMINARY NOTIONS AND ASSUMPTIONS. FORMULATION OF THE MAIN RESULTS.

Let \mathbb{Z}^3 be the three-dimensional hypercubic lattice and

$$\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3 = (-\pi, \pi]^3$$

be the three-dimensional torus (Brillion zone), the dual group of \mathbb{Z}^3 .

Note that operations addition and multiplication by number of the elements of torus $\mathbb{T}^3 \equiv (-\pi, \pi]^3 \subset \mathbb{R}^3$ is defined as operations in \mathbb{R}^3 by the module $(2\pi\mathbb{Z}^3)$.

Let $L^2(\mathbb{T}^3)$ be the Hilbert space of square-integrable functions defined on the torus \mathbb{T}^3 and \mathbb{C}^1 be one-dimensional complex Hilbert space.

We consider a family of the Generalized Friedrichs models acting in $L^2(\mathbb{T}^3)$ as follows:

$$H_\mu(p) = H_0(p) + \mu\Phi^*\Phi, \quad \mu > 0.$$

Here

$$\begin{aligned} \Phi : L^2(\mathbb{T}^3) &\rightarrow \mathbb{C}^1, & \Phi f &= (f, \varphi)_{L^2(\mathbb{T}^3)}, \\ \Phi^* : \mathbb{C}^1 &\rightarrow L^2(\mathbb{T}^3), & \Phi^* f_0 &= \varphi(q)f_0, \end{aligned}$$

where $(\cdot, \cdot)_{L^2(\mathbb{T}^3)}$ – inner product in $L^2(\mathbb{T}^3)$ and $H_0(p)$, $p \in \mathbb{T}^3$ is a multiplication operator by a function $w_p(\cdot) := w(p, \cdot)$, i.e.

$$(1.1) \quad (H_0(p)f)(q) = w_p(q)f(q), \quad f \in L^2(\mathbb{T}^3).$$

Note that for any $f \in L^2(\mathbb{T}^3)$ and $g_0 \in \mathbb{C}^1$ the equality

$$(\Phi f, g_0)_{\mathbb{C}^1} = (f, \Phi^* g_0)_{L^2(\mathbb{T}^3)}$$

holds. The following assumption will be needed throughout the paper.

Hypothesis 1.1. *We assume that the following assumptions are satisfied:*

- (i) *the function $\varphi(\cdot)$ is nontrivial, real-analytic function on \mathbb{T}^3 ;*
- (ii) *the function $w(\cdot, \cdot)$ is real-analytic function on $(\mathbb{T}^3)^2 = \mathbb{T}^3 \times \mathbb{T}^3$ and has a unique non degenerated maximum at $(p_0, q_0) \in (\mathbb{T}^3)^2$.*

The perturbation $v = \Phi^*\Phi$ is positive operator of rank 1. Consequently, by the well-known Weyl theorem [20] the essential spectrum fills the following segment on the real axis:

$$\sigma_{ess}(H_\mu(p)) = \sigma_{ess}(H_0(p)) = [m(p), M(p)],$$

where

$$m(p) = \min_{q \in \mathbb{T}^3} w_p(q), \quad M(p) = \max_{q \in \mathbb{T}^3} w_p(q).$$

By Hypothesis 1.1 there exist such δ -neighborhood $U_\delta(p_0) \subset \mathbb{T}^3$ of the point $p = p_0 \in \mathbb{T}^3$ and analytic vector function $\mathbf{q}_0 : U_\delta(p_0) \rightarrow \mathbb{T}^3$ that for any $p \in U_\delta(p_0)$ the point $\mathbf{q}_0(p) = (q_0^{(1)}(p), q_0^{(2)}(p), q_0^{(3)}(p)) \in \mathbb{T}^3$ is a unique non degenerated maximum of the function $w_p(\cdot)$ (see Lemma 2.1

Moreover, the following integral

$$\frac{1}{\mu(p)} = \int_{\mathbb{T}^3} \frac{\varphi^2(s)ds}{M(p) - w_p(s)} > 0$$

exists (see Lemma 2.4).

The positive number $\mu(p) > 0$ is called *coupling constant threshold*.

Definition 1.2. *The threshold $z = M(p)$ is called a regular point of the essential spectrum of the operator $H_\mu(p)$, if the equation $H_\mu(p)f = M(p)f$ has only trivial solution $f \in L^2(\mathbb{T}^3)$.*

Let $L^1(\mathbb{T}^3)$ be the Banach space of integrable functions on \mathbb{T}^3 .

Definition 1.3. The threshold $z = M(p)$ is called a $M(p)$ energy resonance (virtual level) of the essential spectrum of the operator $H_\mu(p)$, if the equation $H_\mu(p)f = M(p)f$ has a non-trivial solution $f \in L^1(\mathbb{T}^3) \setminus L^2(\mathbb{T}^3)$. The solution f is called resonance state of the operator $H_\mu(p)$.

Remark 1.4. The set \mathbb{G} of $\mu > 0$, for which the threshold is a regular point of the essential spectrum $\sigma_{ess}(H_\mu(p))$ of $H_\mu(p)$, is an open set in $(0, +\infty)$. More precisely, $\mathbb{G} = (0, +\infty) \setminus \{\mu(p)\}$.

Remark 1.5. If the threshold $z = M(p)$ is a regular point of $H_\mu(p)$ then the number of eigenvalues of the operator $H_\mu(p)$ above the threshold $M(p)$ does not change under small perturbations of $\mu \in \mathbb{G}$ (see items (i), (ii) and (iii) of Theorem 1.6).

In the following theorem we have found a necessary and sufficient conditions for existence of a unique eigenvalue $E(\mu, p)$, lying above the threshold of the essential spectrum of $H_\mu(p)$, $p \in U_\delta(p_0)$. We prove that for a fixed $p \in U_\delta(p_0)$, the function $E(\cdot, p)$ is analytic in $(\mu(p), +\infty)$. Moreover for the associated eigenfunction an explicit expression is found and its analyticity is proven. Furthermore, in the case $\mu = \mu(p) > 0$, it is proven that the threshold $M(p)$ of the essential spectrum is either a $M(p)$ energy resonance or eigenvalue for the operator $H_\mu(p)$, $p \in \mathbb{T}^3$.

Theorem 1.6. Assume Hypothesis 1.1 and $p \in U_\delta(p_0)$. Then the following assertions are true.

- (i) The operator $H_\mu(p)$ has a unique eigenvalue $E(\mu, p)$ lying above the threshold $M(p)$ of the essential spectrum if and only if $\mu > \mu(p)$. The function $E(\cdot, p)$ is monotonously increasing real-analytic function in the interval $(\mu(p), +\infty)$ and the function $E(\mu, \cdot)$ is real-analytic in $U_\delta(p_0)$. The associated eigenfunction

$$\Psi(\mu; p, q, E(\mu, p)) = \frac{C\mu\varphi(q)}{E(\mu, p) - w_p(q)}$$

is analytic on \mathbb{T}^3 , where $C \neq 0$ is normalization factor. Moreover, the mappings

$$\Psi : U_\delta(p_0) \rightarrow L^2(\mathbb{T}^3), \quad p \mapsto \Psi(\mu; p, q, E(\mu, p)) \in L^2(\mathbb{T}^3)$$

and

$$\Psi : (\mu(p), +\infty) \rightarrow L^2(\mathbb{T}^3), \quad \mu \mapsto \Psi(\mu; p, q, E(\mu, p)) \in L^2(\mathbb{T}^3)$$

are vector-valued analytic functions in $U_\delta(p_0)$ and $(\mu(p), +\infty)$, respectively.

- (ii) The operator $H_\mu(p)$ has none eigenvalue in semi-infinite interval $(M(p), \infty)$ if and only if $0 < \mu < \mu(p)$.
- (iii) The threshold $z = M(p)$ is a regular point of the operator $H_\mu(p)$ if and only if $\mu \neq \mu(p)$.
- (iv) The threshold $z = M(p)$ is a $M(p)$ energy resonance of the operator $H_\mu(p)$ if and only if $\mu = \mu(p)$ and $\varphi(\mathbf{q}_0(p)) \neq 0$. The associated resonance state is of the form

$$f(q) = \frac{C\mu(p)\varphi(q)}{M(p) - w_p(q)},$$

where $C \neq 0$ is a normalizing constant and $f \in L^1(\mathbb{T}^3) \setminus L^2(\mathbb{T}^3)$.

- (v) The threshold $z = M(p)$ is an eigenvalue of the operator $H_\mu(p)$ if and only if $\mu = \mu(p)$ and $\varphi(\mathbf{q}_0(p)) = 0$. Moreover, if the threshold $z = M(p)$ is an eigenvalue of the operator $H_\mu(p)$ then the associated eigenfunction is of the form

$$(1.2) \quad f(q) = \frac{C\mu(p)\varphi(q)}{M(p) - w_p(q)} \in L^2(\mathbb{T}^3),$$

where $C \neq 0$ is a normalizing constant.

Remark 1.7. From the positivity of $\Phi^* \Phi$ it follows that the operator $H_\mu(p)$ has none eigenvalue lying below $m(p)$.

2. PROOF OF THE RESULTS

We postpone the proof of the theorem after several lemmas and remarks.

Lemma 2.1. Assume Hypothesis 1.1. Then there exist such δ -neighborhood $U_\delta(p_0) \subset \mathbb{T}^3$ of the point $p = p_0$ and analytic function $\mathbf{q}_0 : U_\delta(p_0) \rightarrow \mathbb{T}^3$ that for any $p \in U_\delta(p_0)$ the point $\mathbf{q}_0(p)$ is a unique non-degenerated maximum of the function $w_p(\cdot)$.

Proof. By Hypothesis 1.1 the square matrix

$$A(0) = \left(\frac{\partial^2 w_{p_0}}{\partial q_i \partial q_j}(\mathbf{q}_0) \right)_{i,j=1}^3 < 0$$

is negatively defined and $\nabla w_{p_0}(q_0) = 0$. Then by the implicit function theorem (the analytic case) there exist a δ -neighborhood $U_\delta(p_0) \subset \mathbb{T}^3$ of $p = p_0 \in \mathbb{T}^3$ and a unique analytic vector function $\mathbf{q}_0(\cdot) : U_\delta(p_0) \rightarrow \mathbb{T}^3$ such that $\nabla w_p(q_0(p)) = 0$ and

$$A(p) = \left(\frac{\partial^2 w_p}{\partial q_i \partial q_j}(\mathbf{q}_0(p)) \right)_{i,j=1}^3 < 0, \quad p \in U_\delta(p_0).$$

Hence for any $p \in U_\delta(p_0)$ the point $\mathbf{q}_0(p)$ is a unique non degenerated maximum of the function $w_p(\cdot)$. \square

For any $\mu > 0$ and $p \in \mathbb{T}^3$ we define in $\mathbb{C} \setminus [m(p); M(p)]$ an analytic function $\Delta(\mu, p; \cdot)$ (the Fredholm determinant $\Delta(\mu, p; \cdot)$, associated to the operator $H_\mu(p)$) as

$$(2.1) \quad \Delta(\mu, p; \cdot) = 1 - \mu \Omega(p; \cdot),$$

where

$$(2.2) \quad \Omega(p; z) = \int_{\mathbb{T}^3} \frac{\varphi^2(s) ds}{z - w_p(s)}, \quad p \in \mathbb{T}^3, \quad z \in \mathbb{C} \setminus [m(p); M(p)].$$

Lemma 2.2. A number $z \in \mathbb{C} \setminus \sigma_{ess}(H_\mu(p))$, $p \in \mathbb{T}^3$ is an eigenvalue of the operator $H_\mu(p)$ if and only if $\Delta(\mu, p; z) = 0$. The associated eigenfunction $f \in L^2(\mathbb{T}^3)$ is of the form

$$(2.3) \quad f(q) = \frac{C \mu \varphi(q)}{z - w_p(q)},$$

where $C \neq 0$ is a normalizing constant.

Proof. If a number $z \in \mathbb{C} \setminus \sigma_{ess}(H_\mu(p))$, $p \in \mathbb{T}^3$ is an eigenvalue of the operator $H_\mu(p)$ and $f \in L^2(\mathbb{T}^3)$ is an associated eigenfunction, i.e., the equation

$$(2.4) \quad [\omega_p(q) - z]f(q) - \mu \varphi(q) \int_{\mathbb{T}^3} \varphi(t) f(t) dt = 0,$$

with

$$\int_{\mathbb{T}^3} \varphi(t) f(t) dt \neq 0.$$

has solution, then the solution f of equation (2.4) is given by

$$(2.5) \quad f(q) = \frac{C \mu \varphi(q)}{z - w_p(q)},$$

where $C \neq 0$ is a normalizing constant. The representation (2.5) of the solution of equation (2.4) implies that $\Delta(\mu, p; z) = 0$.

Conversely. Let $\Delta(\mu, p; z) = 0$ for some $z \in \mathbb{C} \setminus \sigma_{ess}(H_\mu(p))$, $p \in \mathbb{T}^3$. Then the function f , defined by (2.5), belong to $L^2(\mathbb{T}^3)$ and obeys the equation $H_\mu(p)f = zf$.

The analyticity of the eigenfunction $f(\cdot)$ defined by (2.5) follows from the analyticity of $\varphi(\cdot)$ and $w_p(\cdot)$ as well as due to the fact that the denominator $z - w_p(\cdot)$ in (2.5) is not vanished. \square

Proposition 2.3. *For $\zeta < 0$ the following equalities hold:*

$$I_n(\zeta) = \int_0^\delta \frac{r^{2n} dr}{r^2 - \zeta} = \frac{\pi}{2} \cdot \frac{\zeta^n}{\sqrt{-\zeta}} + \tilde{I}_n(\zeta), \quad n = 0, 1, 2, \dots,$$

where $\tilde{I}_n(\zeta)$ is an analytic function in a neighborhood of the origin [10].

Lemma 2.4. *Assume Hypothesis 1.1. Then for any $p \in U_\delta(p_0)$ the integral*

$$\Omega(p) = \Omega(p, M(p)) = \int_{\mathbb{T}^3} \frac{\varphi^2(s) ds}{M(p) - w_p(s)}$$

exists and defines an analytic function in $U_\delta(p_0)$.

Proof. We represent the function

$$\Omega(p, z) = \int_{\mathbb{T}^3} \frac{\varphi^2(s) ds}{z - w_p(s)}$$

in the form

$$(2.6) \quad \begin{aligned} \Omega(p, z) &= \int_{U(\mathbf{q}_0(p))} \frac{\varphi^2(s) ds}{z - w_p(s)} + \int_{\mathbb{T}^3 \setminus U(\mathbf{q}_0(p))} \frac{\varphi^2(s) ds}{z - w_p(s)} = \\ &= \Omega_1(p, z) + \Omega_2(p, z), \end{aligned}$$

where $U(\mathbf{q}_0(p))$ is a neighborhood of $\mathbf{q}_0(p)$.

Observe that by Hypothesis 1.1 for any $p \in U_\delta(p_0)$ the function $\Omega_2(p, z)$ is analytic at $z = M(p)$.

We note that by the parametrical Morse lemma for any $p \in U_\delta(p_0)$ there exists a map $s = \psi(y, p)$ of the sphere $W_\gamma(0) \subset \mathbb{R}^3$ with radius $\gamma > 0$ and center at $y = 0$ to a neighborhood $U(\mathbf{q}_0(p))$ of the point $\mathbf{q}_0(p)$ that in $U(\mathbf{q}_0(p))$ the function $w_p(\psi(y, p))$ can be represented as

$$w_p(\psi(y, p)) = M(p) - y_1^2 - y_2^2 - y_3^2 = M(p) - y^2.$$

Here the function $\psi(y, \cdot)$ (resp. $\psi(\cdot, p)$) is analytic in $U_\delta(p_0)$ (resp. $W_\gamma(0)$) and $\psi(0, p) = \mathbf{q}_0(p)$. Moreover, the Jacobian $J(\psi(y, p))$ of the mapping $s = \psi(y, p)$ is analytic in $W_\gamma(0)$ and positive, i.e., $J(\psi(y, p)) > 0$ for all $y \in W_\gamma(0)$ and $p \in U_\delta(p_0)$.

In the integral for $\Omega_1(p, z)$ changing of variables $s = \psi(y, p)$ gives

$$(2.7) \quad \Omega_1(p, z) = \int_{W_\gamma(0)} \frac{\varphi^2(\psi(y, p))}{y^2 + z - M(p)} J(\psi(y, p)) dy,$$

where $J(\psi(y, p))$ is the Jacobian of the mapping $\psi(y, p)$.

Passing to spherical coordinates as $y = r\nu$, we obtain

$$(2.8) \quad \Omega_1(p, z) = \int_0^\gamma \frac{r^2}{r^2 + z - M(p)} \left\{ \int_{\Omega_3} \varphi^2(\psi(r\nu, p)) J(\psi(r\nu, p)) d\nu \right\} dr,$$

where Ω_3 is a unit sphere in \mathbb{R}^3 and $d\nu$ – its element. Inner integral can be represented as

$$(2.9) \quad \int_{\Omega_3} \varphi^2(\psi(r\nu, p)) J(\psi(r\nu, p)) d\nu = \sum_{n=0}^{\infty} \tau_n(p) r^{2n},$$

where $\tau_n(p)$, $n = 0, 1, 2, \dots$ are Pizetti coefficients.

Thus we have that

$$(2.10) \quad \Omega_1(p, z) = \sum_{n=0}^{\infty} \tau_n(p) \int_0^\gamma \frac{r^{2n+2} dr}{r^2 + z - M(p)}, \quad \tau_0(p) = \varphi^2(\mathbf{q}_0(p)) J(\mathbf{q}_0(p))$$

Since the function under the integral sign in (2.9) is analytic in $U_\delta(p_0)$, the Pizetti coefficients $\tau_n(p)$, $n = 0, 1, 2, \dots$ are analytic in $U_\delta(p_0)$. The representation (2.10) yields that the following limit exists

$$\Omega_1(p) = \lim_{z \rightarrow M(p)+0} \Omega_1(p, z) = \lim_{z \rightarrow M(p)+0} \sum_{n=0}^{\infty} \tau_n(p) \int_0^\gamma \frac{r^{2n+2} dr}{r^2 + z - M(p)} = \sum_{n=0}^{\infty} \frac{\gamma^{2n+1}}{2n+1} \tau_n(p)$$

and consequently,

$$(2.11) \quad \Omega(p) = \lim_{z \rightarrow M(p)+0} \Omega(p, z) = \Omega_1(p) + \Omega_2(p),$$

where $\Omega_2(p) = \Omega_2(p, M(p))$. The analyticity of the Pizetti coefficients $\tau_n(p)$, $n = 0, 1, 2, \dots$ in $U_\delta(p_0)$ yield that the function $\Omega_1(p)$ is analytic in $p \in U_\delta(p_0)$. So, $\Omega(p)$ is analytic in $p \in U_\delta(p_0)$. \square

Lemma 2.5. Assume Hypothesis 1.1 and $p \in U_\delta(p_0)$. Then the following statements are equivalent:

(i) the threshold $M(p)$ is a resonance of the operator $H_\mu(p)$ and the associated resonance state is of the form

$$(2.12) \quad f(q) = \frac{C\mu(p)\varphi(q)}{M(p) - w_p(q)},$$

where $C \neq 0$ is a normalizing constant.

(ii) $\varphi(\mathbf{q}_0(p)) \neq 0$ and $\Delta(\mu, p; M(p)) = 0$.

(iii) $\varphi(\mathbf{q}_0(p)) \neq 0$ and $\mu = \mu(p)$.

Proof. Let the threshold $M(p)$ be a resonance of the operator $H_\mu(p)$. According to the definition of resonance the equation $H_\mu(p)f = M(p)f$ has a nontrivial solution $f \in L^1(\mathbb{T}^3) \setminus L^2(\mathbb{T}^3)$, i.e., the equation

$$(2.13) \quad [M(p) - w_p(q)]f(q) - \mu\varphi(q) \int_{\mathbb{T}^3} \varphi(t)f(t)dt = 0,$$

with

$$\int_{\mathbb{T}^3} \varphi(t)f(t)dt \neq 0.$$

has a nontrivial solution. It is easy to check that the solution f of equation (2.13), i.e., the resonance state, is given by (2.12). Since $w_p(\cdot)$ has a unique non-degenerate maximum at $\mathbf{q}_0(p) \in \mathbb{T}^3$, in the integral

$$\Omega_1(p) = \int_{W_\gamma(0)} \frac{\varphi^2(\psi(y, p))J(\psi(y, p))dy}{y^4}.$$

passing to spherical coordinates as $y = r\nu$ we get

$$(2.14) \quad \Omega_1(p) = \int_0^\gamma \left(\int_{\Omega_3} \varphi^2(\psi(r\nu, p))J(\psi(r\nu, p)) d\nu \right) r^{-2} dr.$$

Expanding the function $\varphi(\psi(r\nu, p))$ to the Taylor series at $r = 0$ we obtain

$$(2.15) \quad \varphi(\psi(r\nu, p)) = \varphi(\mathbf{q}_0(p)) + \sum_{i=1}^3 \frac{\partial \varphi}{\partial \psi^{(i)}}(\mathbf{q}_0(p)) \left(\sum_{j=1}^3 \frac{\partial \psi^{(i)}}{\partial y_j}(0, p) \nu_j \right) r + g(r, \nu)r^2, \quad y_j = r\nu_j,$$

where $g(\cdot, \nu)$ is continuous in $W_\gamma(0)$ and $\nu_1^2 + \nu_2^2 + \nu_3^2 = 1$. Since the solution f of the equation (2.13) belongs to $L^1(\mathbb{T}^3) \setminus L^2(\mathbb{T}^3)$ the asymptotics (2.15) yields the relation $\varphi(\mathbf{q}_0(p)) \neq 0$.

Putting the expression (2.12) for f to the equation (2.13) yields

$$(2.16) \quad \varphi(q) - \mu\varphi(q) \int_{\mathbb{T}^3} \frac{\varphi^2(t)dt}{M(p) - w_p(t)} = 0,$$

which implies the equalities $\Delta_\mu(p, M(p)) = 0$, and $\mu = \mu(p)$.

Let $\varphi(\mathbf{q}_0(p)) \neq 0$ and $\mu = \mu(p)$. Then it easy to check that $\Delta(\mu, p; M(p)) = 0$ and the function f , defined by (2.12), belongs to $L^1(\mathbb{T}^3) \setminus L^2(\mathbb{T}^3)$ and obeys the equation $H_\mu(p)f = M(p)f$. □

Lemma 2.6. *Assume Hypothesis 1.1 and $p \in U_\delta(p_0)$. Then the following statements are equivalent:*

- (i) *The threshold $z = M(p)$ is an eigenvalue of the operator $H_\mu(p)$ and the associated eigenvector is of the form*

$$(2.17) \quad f(q) = \frac{C\mu(p)\varphi(q)}{M(p) - w_p(q)},$$

where $C \neq 0$ is a normalizing constant.

- (ii) *$\varphi(\mathbf{q}_0(p)) = 0$ and $\Delta(\mu, p; M(p)) = 0$.*

- (iii) *$\varphi(\mathbf{q}_0(p)) = 0$ and $\mu = \mu(p)$.*

Proof. Let $z = M(p)$ be an eigenvalue of the operator $H_\mu(p)$ and $f \in L^2(\mathbb{T}^3)$ is an associated eigenfunction, i.e., the equation

$$(2.18) \quad [M(p) - w_p(q)]f(q) - \mu\varphi(q) \int_{\mathbb{T}^3} \varphi(t)f(t)dt = 0,$$

with

$$\int_{\mathbb{T}^3} \varphi(t) f(t) dt \neq 0,$$

has nontrivial solution. Then the associated eigenfunction f is given by (2.17). In this case the relation $f \in L^2(\mathbb{T}^3)$ and asymptotics (2.15) yield the equality $\varphi(\mathbf{q}_0(p)) = 0$. The equation 2.2 implies the equality $\Delta(\mu, p; M(p)) = 0$, which yields that $\mu = \mu(p)$.

Let $\varphi(\mathbf{q}_0(p)) = 0$ and $\mu = \mu(p)$. Then $\Delta(\mu, p; M(p)) = 0$ and the function f , defined by (2.17), obeys the equation $H_\mu(p)f = M(p)f$. \square

Corollary 2.7. *The equation $H_\mu(p)f = M(p)f$ has only trivial solution $f = 0 \in L^2(\mathbb{T}^3)$ if and only if $\mu \neq \mu(p)$.*

In the following lemma we establish an expansion for $\Delta(\mu, p; z)$ in a half-neighborhood $(M(p), M(p) + \delta)$ of the point $z = M(p)$.

Lemma 2.8. *Assume Hypothesis 1.1. Then for any $\mu > 0$, $p \in U_\delta(p_0)$ and sufficiently small $z - M(p) > 0$ the function $\Delta(\mu, p; \cdot)$ can be represented as following convergent series*

$$(2.19) \quad \Delta(\mu, p; z) = 1 - \mu\Omega(p) + \mu \frac{\pi\tau_0(p)}{2} (z - M(p))^{1/2} - \mu \sum_{n=2}^{\infty} c_n(p) (z - M(p))^{n/2},$$

$$\tau_0(p) = \varphi^2(\mathbf{q}_0(p)) J(\mathbf{q}_0(p)),$$

Proof. According to (2.10) and Proposition 2.3 the function $\Omega_1(p; z)$ can be written as

$$(2.20) \quad \Omega_1(p, z) = -\frac{\pi\tau_0(p)}{2} (z - M(p))^{1/2} + \sum_{n=1}^{\infty} \tilde{c}_n(p) (z - M(p))^{n+1/2} + \tilde{F}(p, z),$$

where $\tilde{F}(p, z)$ is an analytic function at the point $z = M(p)$ and

$$\tilde{c}_n(p) = \frac{(-1)^{n+1} \pi \tau_n(p)}{2}.$$

Consequently, the decomposition (2.6) yields for $\Omega(p, z)$, $z \in [M(p), M(p) + \delta)$ the following representation

$$\Omega(p, z) = -\frac{\pi\tau_0(p)}{2} (z - M(p))^{1/2} + \sum_{n=1}^{\infty} \tilde{c}_n(p) (z - M(p))^{n+\frac{1}{2}} + F(p, z),$$

where $F(p, z) = \tilde{F}(p, z) + \Omega_2(p, z)$ is analytic function at the point $z = M(p)$.

Notifying $F(p, M(p)) = \Omega(p)$ and $(z - M(p))^{1/2} > 0$ for $z > M(p)$, we obtain

$$\Omega(p, z) = \Omega(p) - \frac{\pi\tau_0(p)}{2} (z - M(p))^{1/2} + \sum_{n=2}^{\infty} c_n(p) (z - M(p))^{n/2}.$$

The equality (2.1) proves Lemma 2.8. \square

Now we prove the main results.

Proof of Theorem 1.6. (i) Let $\mu > \mu(p)$. Then Lemma 2.8 gives that

$$\lim_{z \rightarrow M(p)+0} \Delta(\mu, p; z) = \Delta(\mu, p; M(p)) = 1 - \frac{\mu}{\mu(p)} < 0.$$

The function $\Delta(\mu, p; \cdot)$ is continuous and monotonously increasing in $z \in (M(p), +\infty)$ and

$$(2.21) \quad \lim_{z \rightarrow +\infty} \Delta(\mu, p; z) = 1.$$

Whence, $\Delta(\mu, p; z) = 0$ for a unique $z \in (M(p), +\infty)$.

Let $\Delta(\mu, p; z) = 0$ for some $z \in (M(p), +\infty)$. Then

$$(2.22) \quad 1 - \frac{\mu}{\mu(p)} = \Delta(\mu, p; M(p)) < \Delta(\mu, p; z) = 0$$

which yields that $\mu > \mu(p)$. The Lemma 2.6 ended the proof of the statement.

Since $z = E(\mu, p)$ is a solution of the equation $\Delta(\mu, p; z) = 0$ and $\Delta(\mu, \cdot; z)$ (resp. $\Delta(\cdot, p; z)$) is real-analytic in $U_\delta(p_0)$ (resp. $(\mu(p), +\infty)$), the implicit function theorem implies that $E(\mu, \cdot)$ (resp. $E(\cdot, p)$) is real analytic in $U_\delta(p_0)$ (resp. $(\mu(p), +\infty)$).

Note that $p \in U_\delta(p_0)$ the function $\Delta(\cdot, p; z)$ monotonously decreases in $(\mu(p), +\infty)$ and hence the solution (eigenvalue) $E(\mu, p)$ also monotonously decreases in $(\mu(p), +\infty)$.

Lemma 2.2 implies that if the number $E(\mu, p)$ is an eigenvalue of $H_\mu(p)$, $p \in U_\delta(p_0)$, then the function

$$\Psi(\mu; p, \cdot, E(\mu, p)) = \frac{C\mu\varphi(\cdot)}{E(\mu, p) - w_p(\cdot)},$$

where $C \neq 0$ is a normalization constant, is a solution of the equation

$$H_\mu(p)\Psi(\mu; p, q, E(\mu, p)) = E(\mu, p)\Psi(\mu; p, q, E(\mu, p)).$$

The analyticity of $\Psi(\mu; p, \cdot, E(\mu, p))$ follows from the analyticity of $\varphi(\cdot)$ and $(w_p(\cdot) - E(\mu, p))^{-1}$ in \mathbb{T}^3 .

Since the functions $E(\mu, \cdot)$ (resp. $E(\cdot, p)$) and $w(\cdot, q)$ are analytic in $U_\delta(p_0)$ (resp. $(\mu(p), +\infty)$) and $w_p(q) - E(\mu, p) > 0$ the mapping $p \mapsto \Psi(\mu; p, q, E(\mu, p))$ (resp. $\mu \mapsto \Psi(\mu; p, q, E(\mu, p))$) is also analytic mapping in $U_\delta(p_0)$ (resp. $(\mu(p), +\infty)$).

We can prove the rest part of statements of Theorem 1.6 applying Lemmas 2.5 and 2.6 by the same way as the proof of (i). \square

3. ACKNOWLEDGMENTS

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REFERENCES

- [1] ZH. I. ABDULLAEV, I. A. IKROMOV, AND S. N. LAKAEV: Theor. Math. Phys., 103, 390397 (1995).
- [2] S. ALBEVERIO, S. N. LAKAEV, Z. I. MUMINOV: The threshold effects for a family of Generalized Friedrichs models under rank one perturbations. J. Math. Anal. Appl. **330** (2007), no. 2, 1152 – 1168.
- [3] S. ALBEVERIO, S. N. LAKAEV, Z. I. MUMINOV: The Threshold Effects for the Two-particle Hamiltonians on Lattices, Comm.Math.Phys. **262**(2006), 91–115.
- [4] R. COURANT, D. HILBERT: Methods of Mathematical Physics. Vol. 2, Partial Differential Equations. Wiley Classics Edition Published in 1989.
- [5] J. HECKER DENSCHLAG, A. J. DALEY: Exotic atom pairs: Repulsively bound states in an optical lattice, arXiv:cond-mat/0610393 [cond-mat.other]
- [6] K. O. FRIEDRICHS: On the perturbation of continuous spectra. Communications on Appl. Math. **1**(1948), 361–406.
- [7] K. O. FRIEDRICHS: Perturbation of Spectra in Hilbert Space American Mathematical Society, 2008.
- [8] M. KLAUS, B. SIMON: Coupling Constant Thresholds in Nonrelativistic Quantum Mechanics. I. Short-Range Two-Body Case. Annals of Physics **130**, 251–281, New Jersey, 1980. of the Generalized Friedrichs model, Funct. Anal. Appl. **17** (1983), 317–331

- [9] S.N.LAKAEV: Some spectral properties of the Generalized Friedrichs model. Trudy Sem. Petrovsk. **11** (1986), 210–238; English transl.in J.Soviet Math. **45** (1989), no 6
- [10] S.N.LAKAEV: Bound states and resonances of N -particle discrete Schrödinger operators. Theor. and Math. Phys. **91**(1992), No.1, 51-65.
- [11] S.N. LAKAEV, A.IBRAHIM, SH.KURBANOV: Threshold Effects for the Generalized Generalized Friedrichs Model with the Perturbation of Rank One. Abstract and Applied Analysis **2012** (2012), Article ID 180953 (14 pages).
- [12] S.N. LAKAEV, M. DARUS, SH.H. KURBANOV: Puiseux series expansion for an eigenvalue of the generalized Generalized Friedrichs model with perturbation of rank 1. J. Phys. A: Math. Theor. **46** **2013**(2013), 000000 (15pp).
- [13] S.N. LAKAEV, SH.YU. HOLMATOV: Asymptotics of Eigenvalues of a two-particle Schrödinger operators on lattices with zero range interaction, J. Phys. A: Math. Theor. **44**(2011), 135304,19 pp.
- [14] S.N.LAKAEV: The stucture of resonances of the Generalized Generalized Friedrichs model, Funct. Anal. Appl. **17** (1983), 317–331.
- [15] S.N.LAKAEV: Some spectral properties of the Generalized Friedrichs model, Trudy Sem. Petrovsk. **11** (1986), 210–238; English transl.in J.Soviet Math. **45** (1989), no 6.
- [16] E.L.LAKSHTANOV, R.A.MINLOS: The spectrum of two-particle bound states of transfer matrices of Gibbs fields (an isolated bound state). (Russian) Funktsional. Anal. i Prilozhen. **38**(2004), No.3, 52–69; (translation in Funct. Anal. Appl. **38** (2004), No. 3, 202–216).
- [17] V.A.MALISHEV, R.A.MINLOS: Linear infinite-particle operators. Translations of Mathematical Monographs, **143**(1995) American Mathematical Society, Providence, RI.
- [18] A. MOGILNER: Hamiltonians in solid state physics as multi-particle discrete Schrödinger operators: Problems and results. Advances in Soviet Mathematics **5**, 139–194 (1991).
- [19] M.REED, B.SIMON: Methods of modern mathematical physics. III: Scattering theory, Academic Press, New York, 1979.
- [20] M. REED and B. SIMON: Methods of modern mathematical physics. IV: Analysis of operators. Academic Press, N.Y., 1978.
- [21] K. WINKLER, G. THALHAMMER, F. LANG, R. FRIMM J. H. DENSCHLAG, A. J. DALEY, A. KANTIAN, H. P. BÜCHLER and P. ZOLLER: Repulsively bound atom pairs in an optical lattice. LETTERS. Vol. 441|15 June 2006| doi:10.1038/nature04918 nature